M.A./M.Sc. Sem II 9th Paper

TOPOLOGY

Unit 5 (Contd.)

Q. Prove that every T_3-space is a T_2-space.

Soln. Let X be a T_3-space.

$\Rightarrow$ By definition of T_3-space, X is a regular T_1-space.

$\because$ Every singleton subset of a T_1-space is a closed set.

Let x, y be two distinct points of X, i.e.

$x, y \in X$ and $x \neq y$.

$\because$ X is a T_1-space $\Rightarrow$ $\{x\}$ being a singleton subset, is closed.

$\because$ $y \neq x \Rightarrow y \notin \{x\}$.

$\because$ X is a regular space,

$\Rightarrow$ by definition $\exists$ disjoint open sets G and H such that

$\{x\} \subseteq G$ and $y \in H$.

$\Rightarrow x \in G$, $y \in H$ and $G \cap H = \phi$

Thus, x and y belong to disjoint open sets G and H respectively.

$\Rightarrow$ X is a T_2-space. Proved

Q. A topological space X is regular iff for every $x \in X$ and for every open set G containing x, there is an open set H such that

$$x \in H \subset \overline{H} \subset G.$$

Proof

Necessary part

Let X be a regular space. Let $x \in X$.

Let G be an open set containing x.

$\Rightarrow x \in G \quad \Rightarrow x \notin G^c$ and G^c is closed.

So, there exist disjoint open sets H and W such that

$$x \in H, \; G^c \subset W$$

But $H \cap W = \phi \Rightarrow H \subseteq W^c$

$$\Rightarrow H \subseteq \overline{W^c} = W^c \subset G.$$

Thus $x \in H \subset \overline{H} \subset G$. Proved

Sufficient part

Let $x \in X$ and F be a closed set such that $x \notin F$.

$\Rightarrow F^c$ is an open set containing x.

Given that $\exists$ an open set H such that

$$x \in H \subset \overline{H} \subset F^c$$

Then H and $\overline{H}^c$ are disjoint open sets such that $x \in H$ and $F \subset \overline{H}^c$.

Hence, X is a regular space.

Proved